This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

Stackelberg-Nash game approach for constrained robust optimization with fuzzy variables

Jie Han, Chunhua Yang, *Senior Member, IEEE,* Cheng-Chew Lim, *Senior Member, IEEE,* Xiaojun Zhou, and Peng Shi, *Fellow, IEEE*

***Abstract*—In this paper, the problem of robust optimization is considered for dynamical systems with both constraints and uncertainties. Conditions are established to ensure the existence of solutions to the problem with both robust optimality and feasibility. The objective performance with respect to fuzzy uncertainties is evaluated based on the expectation-entropy model. A feasibility robustness analysis method is proposed to handle the uncertainties in the constraints. Using the hierarchy structure in robust design, the optimization framework based on Stackelberg-Nash game is developed. A leader-followers state transition algorithm is designed to search for the equilibrium solution. Two application examples are given to demonstrate that the proposed robust optimization method can accurately evaluate the robustness performance and successfully search for a compromise solution.**

***Index Terms*—Constrained robust optimization, fuzzy variable, feasibility robustness analysis, Stackelberg-Nash game, state transition algorithm**

I. Introduction

Fuzzy uncertainties are unavoidable in practical engineering problems due to the vagueness of subjective judgment and the impreciseness of objective knowledge [1]. As a typical representative of uncertainty measurement, the robustness has been an important concept in structure design and process optimization.

Robust optimization theory is developed to find solutions that are insensitive to uncertainties. In order to analyze the sensitivity region of candidates, the uncertain parameter has been treated as an interval variable, and the nominal value and the variation value of objectives and constraints have been estimated [2]. This type of method can guarantee that the solution is feasible for all admissible values of uncertain parameters and the objective function has robust performance [3]. However, it is too conservative to obtain a solution that is insensitive to all uncertainties. This leads to a trade-off model (multi-objective model) to balance the robustness and the cost of robustness [4][5]. Here, the robust optimization method makes use of the possibility characteristics of fuzzy uncertainty and optimizes the average performance of objectives and constraints. For solving practical optimization problems, these methods can provide relatively robust solution that is feasible and close to optimal for most of admissible uncertain parameters.

This work was funded by the National Natural Science Foundation of China (Grant Nos. 61860206014, 61873285), the Innovation-Driven Plan in Central South University (Grant No. 2018CX012), and the 111 Project (Grant No. B17048).*(Corresponding author: Xiaojun Zhou.)*

J. Han, C. Yang and X. Zhou are with the School of Automation, Central South University, Changsha 410083, China, and Xiaojun Zhou is also with the Peng Cheng Laboratory, Shenzhen 518000, China. (e-mail: hanjie@csu.edu.cn; ychh@csu.edu.cn; michael.x.zhou@csu.edu.cn).

C.C. Lim and P. Shi are with the School of Electrical and Electronic Engineering, The University of Adelaide, Adelaide 5005, Australia (e-mail: cheng.lim@adelaide.edu.au; peng.shi@adelaide.edu.au).

Manuscript received XX XX, 2020; revised XX XX, 2020.

The trade-off model in a realistic robust optimization considers both optimality robustness and feasibility robustness. For fuzzy objective function, the optimality robustness is evaluated based on the expectation-entropy model [6]. The fuzzy expectation can represent the average performance of the objective function, and the fuzzy entropy can quantify the uncertainty of the objective performance [7]. For fuzzy inequality constraints, the reliability assessment model is established to measure the safety possibility [8]. The traditional reliability analysis treats a solution as safety state only if all its possible constraint values meet the requirements. This operation makes the reliability analysis results too conservative to accurately evaluate constraints performance [9]. Wang et al. modified the fuzzy reliability model based on the interval ranking strategy, and made it more reliable to analyze various fuzzy systems [10]. For fuzzy equality constraints, previous studies focused on linear programming and transformed the fuzzy equality constraints to the crisp ones using the measure of the similarity [11][12]. In order to handle fuzzy constraint in general optimization problem, it is necessary to design a comprehensive analysis method to evaluate the feasibility robustness for inequality constraints and equality constraints.

In game theory, there are three major strategies used when searching for a compromise solution considering both optimality robustness and feasibility robustness [13][14][15][16]: the Pareto-based strategy, the Nash-based strategy, and the Stackelberg-based strategy. Let each player correspond to a metric to be optimized. The Pareto-based strategy is a cooperative game in which players have knowledge of the decisions made by other players. Through communication and cooperation among the players, a set of Pareto optimal solutions can be obtained. This Pareto optimal solution set has no preference for the objective function. Decision makers will have to select the final optimal solution from the Pareto-front. The Nash-based strategy is a non-cooperative game in which players have equal status and act independently. A steady Nash equilibrium will be reached when each player cannot further improve its own outcome due to the constraint of the other players' decision. The Stackelberg-based strategy is also a non-cooperative game in which players have a hierarchy of leaders and followers. The leaders could anticipate the

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

followers' reaction and optimize their decision accordingly, while the followers can only optimize their strategies based on the leaders' decision.

When evaluating the solution quality for a constrained robust optimization problem, feasibility robustness is more important than optimality robustness [17]. In the context of finding a realistic robust solution for the fuzzy optimization problem, the feasibility robustness is regarded as the leader objective and the optimality robustness is taken as the follower objective. Since there are two metrics (expectation and entropy) to evaluate the optimality robustness, the Nash-based strategy can be used to handle the relationship between two followers. Therefore, the Stackelberg-Nash game is investigated to optimize the fuzzy problem with one-leader and two-follower strategies, aiming to find a solution with preference towards feasibility robustness.

The novelty and contribution of this study is threefold: (1) a comprehensive feasibility robustness analysis method is proposed, which includes the possibilistic safety (failure) index for inequality constraints and the discrimination index for equality constraints; (2) a Stackelberg-Nash based robust optimization model is established, which can not only subjectively choose the feasibility robustness as a priority target, but also objectively balance the trade-off between robustness metrics; (3) a leader-followers state transition algorithm is designed to search for the Stackelberg-Nash equilibrium. The algorithm overcomes the nonlinearity and nonconvexity of the players' objective function. Two engineering optimization problems with fuzzy parameters are used to verify the effectiveness of the proposed robust optimization method.

The remainder of this paper is organized as follows. Section 2 describes the robust optimization problem with fuzzy parameters. A comprehensive feasibility robustness analysis method is introduced in Section 3. Section 4 provides the framework of robust optimization method based on the Stackelberg-Nash game. Applications of the robust optimization method in using a vehicle side impact design and a power scheduling design are presented in Section 5. Section 6 concludes this paper and offers some possible future research directions.

II. PROBLEM FORMULATION

Consider a general constrained optimization problem:

Problem P1

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathbb{R}^n} \ f(\boldsymbol{x},\tilde{\boldsymbol{p}}) \\ &\text{s.t. } g_j(\boldsymbol{x},\tilde{\boldsymbol{p}}) \le \tilde{a_j}, j=1,\cdots,q, \\ &\qquad h_j(\boldsymbol{x},\tilde{\boldsymbol{p}}) = \tilde{b_j}, j=1,\cdots,r, \end{aligned}$$

where $f(\boldsymbol{x},\tilde{\boldsymbol{p}})$ is the objective function; $\boldsymbol{x}=(x_1,x_2,\dots,x_n)$ is a n-dimensional decision variable bounded by its lower and upper limits ($\boldsymbol{x}_l \le \boldsymbol{x} \le \boldsymbol{x}_u$); $\tilde{\boldsymbol{p}}$ is the parameter vector, which could be fuzzy variable and uses membership function to represent the degree of uncertainty; $g(\boldsymbol{x},\tilde{\boldsymbol{p}})$ is the set of q inequality constraints and $h(\boldsymbol{x},\tilde{\boldsymbol{p}})$ is the set of r equality constraints; $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ are the right hand side of each constraint, which can also be fuzzy variables.

To analyze the uncertainty in both objective function and constraint functions, in this paper, Problem **P1** will be transformed to a fuzzy robust optimization problem considering both optimality robustness and feasibility robustness.

A. Optimality robustness

Optimality robustness evaluates the sensitivity of the objective function to uncertainties. For (almost) all possible values of uncertain parameters, the objective function value should remain close to the optimal value or have little deviation from the optimal value [4]. Therefore, to handle the fuzzy uncertainties in objective function, the expectation-entropy model is studied to analyze the optimality robustness.

Presented here are some basic definitions of fuzzy variable based on uncertainty theory [7].

Definition 1. Let $\tilde{\beta}$ denote a fuzzy variable with assigned membership function $\mu(t)$ and z be a real number, then the credibility $\text{Cr}\{\cdot\}$ that event "$\tilde{\beta} \le z$" will occur is defined by

$$\text{Cr}\{\tilde{\beta} \le z\} = \frac{1}{2}(\sup_{t\le z}\mu(t) + 1 - \sup_{t>z}\mu(t)). \tag{1}$$

The fuzzy event must hold if its credibility value is 1 and fail if the credibility value is 0.

Definition 2. Let $\tilde{\beta}$ denote a fuzzy variable. The expected value of $\tilde{\beta}$ is defined by

$$E[\tilde{\beta}] = \int_0^{+\infty} \text{Cr}\{\tilde{\beta} \ge z\}\text{d}z - \int_{-\infty}^0 \text{Cr}\{\tilde{\beta} \le z\}\text{d}z. \tag{2}$$

Definition 3. Let $\tilde{\beta}$ denote a fuzzy variable. The entropy of $\tilde{\beta}$ is defined by

$$\begin{aligned} D[\tilde{\beta}] = &\int_{-\infty}^{+\infty} (-\text{Cr}\{\tilde{\beta}=z\}\ln \text{Cr}\{\tilde{\beta}=z\})\text{d}z \\ &+ \int_{-\infty}^{+\infty} (-(1-\text{Cr}\{\tilde{\beta}=z\})\ln(1-\text{Cr}\{\tilde{\beta}=z\}))\text{d}z. \end{aligned} \tag{3}$$

The expected value $E[\cdot]$ represents the average value of a fuzzy variable, and the entropy value $D[\cdot]$ provides a quantitative measure of the uncertainty (variability) associated with fuzzy variable [18]. Based on the fuzzy entropy model, we can estimate the fuzziness resulting from information deficiency caused by the inability to accurately predict the specified values, so as to evaluate the robustness performance of objective functions. The smaller the fuzzy entropy is, the less fuzziness of the variable.

Based on the definitions of fuzzy expectation and fuzzy entropy, the fuzzy objective function in Problem **P1** can be reformulated to evaluate the optimality robustness via:

$$\begin{cases} \min\limits_{\boldsymbol{x}\in\mathbb{R}^n} \quad E[f(\boldsymbol{x},\tilde{\boldsymbol{p}})], & (4) \\ \min\limits_{\boldsymbol{x}\in\mathbb{R}^n} \quad D[f(\boldsymbol{x},\tilde{\boldsymbol{p}})]. & (5) \end{cases}$$

The aim of (4) is to find a solution with the best average performance; and the aim of (5) is to minimize the fuzziness of the performance. With the above two functions, the minimization of the uncertain function $f(\cdot)$ can be transformed to minimize two deterministic quality indexes.

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

B. Feasibility robustness

Feasibility robustness provides a measure of constraints' uncertainty. For (almost) all possible values of uncertain parameters, the solution should remain feasible [4]. In order to evaluate the fuzzy uncertainties of the constraints, the fuzzy reliability based on possibilistic safety index is analyzed.

For inequality constraints, let us consider a constraint $g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \leq a$, whose parameter $\tilde{\boldsymbol{p}}$ is a fuzzy variable and right hand side parameter a is a real number. With the fuzzy parameter $\tilde{\boldsymbol{p}}$, the constraint value $G = g(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ is also a fuzzy variable and its membership function μ_G can be calculated based on α-cut technique [19]. Fig.1 gives an example of the order relation between G and a, where the variable t represents the possible constraint value.

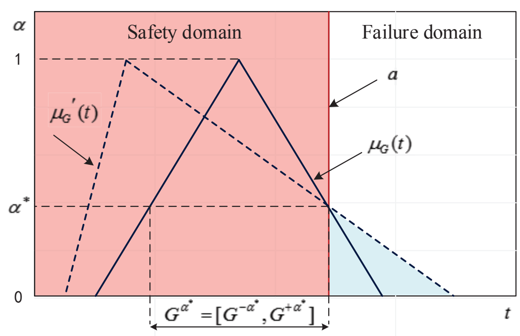


Fig. 1: Order relation among G, G' and a

For any membership level $\alpha \in [0,1]$, the cut set $G^{\alpha} = \{t | \mu_G(t) \geq \alpha\}$ is an interval defined as $[G^{-\alpha}, G^{+\alpha}]$. The critical condition $G = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) = a$ divides the variable space into two parts: the safety domain Ω_s with $g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) < a$ and the failure domain Ω_f with $g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) > a$. If the lower bound of the cut set satisfies $G^{-\alpha} > a$, then all possible values of G at membership level α are bigger than a, and the constraint fails. Conversely, if the upper bound of the cut set satisfies $G^{+\alpha} < a$, then all possible values of G satisfy the constraint and remain safe. For the transition domain with $G^{-\alpha} < a < G^{+\alpha}$, the minimum membership level α^* is defined to represent the safety state [10]:

$$\alpha^* = \inf_{G^{+\alpha} < a, \alpha \in [0,1]} \alpha. \tag{6}$$

Based on the traditional fuzzy reliability theory [9], the interval possibilistic safety index $d_s(\alpha)$ under different membership levels can be defined as

$$d_s(\alpha) = \mathrm{Poss}\{G^{\alpha} = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}^{\alpha}) < a\} = \begin{cases} 0 & \text{if} \quad \alpha < \alpha^* \\ 1 & \text{if} \quad \alpha \geq \alpha^* \end{cases} \tag{7}$$

where $\mathrm{Poss}\{\cdot\}$ represents the possibility of the event $\{\cdot\}$, the interval possibilistic failure index is $d_f(\alpha) = 1 - d_s(\alpha)$. By aggregating all possibilities under different membership levels, the final fuzzy possibilistic safety index Π_s and fuzzy possibilistic failure index Π_f can be calculated as:

$$\Pi_s = \mathrm{Poss}\{G = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \leq a\} = \int_0^1 d_s(\alpha) \mathrm{d}\alpha = 1 - \alpha^*, \tag{8}$$

$$\Pi_f = \mathrm{Poss}\{G = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) > a\} = \int_0^1 d_f(\alpha) \mathrm{d}\alpha = \alpha^*. \tag{9}$$

In order to find a solution with high feasibility robustness, the fuzzy possibilistic safety index should be maximized or the fuzzy possibilistic failure index should be minimized. Therefore, the inequality constraints with fuzzy uncertainties can be transformed to the following deterministic forms:

$$\max_{\boldsymbol{x} \in \mathbb{R}^n} \sum_{i=1}^{q} \Pi_{s,i}[g_i(\boldsymbol{x}, \tilde{\boldsymbol{p}})] \text{ or } \min_{\boldsymbol{x} \in \mathbb{R}^n} \sum_{i=1}^{q} \Pi_{f,i}[g_i(\boldsymbol{x}, \tilde{\boldsymbol{p}})]. \tag{10}$$

It should be noted that the state of constraint value in transition domain is either "safe" or "failure", which ignores additional possibility information. For example, as illustrated in Fig.1, the fuzzy variables G and G' have the same α^*-cut set. According to (8), even if fuzzy variables have different membership functions, their fuzzy possibilistic safety indexes will not stay same. Therefore, the possibilistic safety index in transition state should be modified to accurate qualify the fuzzy information.

To our knowledge, there are few studies on the uncertainty measurement for nonlinear fuzzy equality constraint. Therefore, in next section, we will provide a comprehensive feasibility robustness analysis method for various of fuzzy constraints.

III. Comprehensive Feasibility Robustness Analysis Method

In Problem **P1**, there are inequality constraints and equality constraints, and both sides of the constraint could contain fuzzy parameters. Fig.2 gives six kinds of constraints with fuzzy uncertainties, in which $\tilde{\boldsymbol{p}}$, $\tilde{a}$, and $\tilde{b}$ are fuzzy variables. With fuzzy parameter $\tilde{\boldsymbol{p}}$, the left hand side of the constraint is also an fuzzy variable, and its membership function is assumed to be triangular shape. With fuzzy right hand side parameters $\tilde{a}$ and $\tilde{b}$ (assumed to be triangular fuzzy variables), there are different safety levels for different constraint values. The red color in Fig.2 represents the safety domain, and the darker the color, the safer the constraint value.

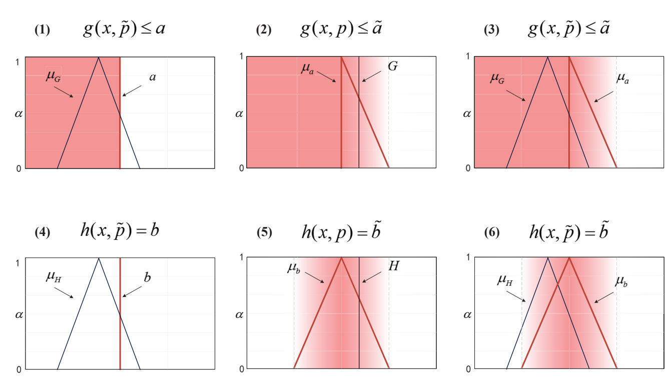


Fig. 2: Description of fuzzy inequality constraint and equality constraint

Based on the characteristics of different constraints in Fig.2, we propose a comprehensive analysis method to evaluate the feasibility robustness.

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

A. Feasibility robustness analysis for inequality constraint

To find a solution with high possibilistic safety index, the traditional fuzzy reliability theory is found to be too conservative to accurately measure the safety index. Thus, an analysis method based on the interval ranking strategy is proposed to evaluate the feasibility robustness for inequality constraint.

Under any membership level $\alpha \in [0,1]$, the fuzzy variable can be transformed to an interval variable. Based on the interval possibility theory [9] [20], the possibility that $G^{\alpha} = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}^{\alpha}) = [G^{-\alpha}, G^{+\alpha}]$ is larger than $\tilde{a}^{\alpha} = [\tilde{a}^{-\alpha}, \tilde{a}^{+\alpha}]$ can be calculated by

$$\text{Poss}\{G^{\alpha} < \tilde{a}^{\alpha}\} = \frac{\max(0, w_{G^{\alpha}} + w_{\tilde{a}^{\alpha}} - \max(0, G^{+\alpha} - \tilde{a}^{-\alpha}))}{w_{G^{\alpha}} + w_{\tilde{a}^{\alpha}}}. \quad (11)$$

where $w_{G^{\alpha}} = G^{+\alpha} - G^{-\alpha}$ and $w_{\tilde{a}^{\alpha}} = \tilde{a}^{+\alpha} - \tilde{a}^{-\alpha}$ On this basis, the possibilistic index for inequality constraints in Fig.2 (cases 1, 2, and 3) can be analyzed as:

- Case 1: the left hand side parameter ($\tilde{\boldsymbol{p}}$) is a fuzzy number, and the right hand side parameter (a) is a real number. We only accept constraint value less than a as a safe state. The possibilistic safety index and possibilistic failure index are modified to:

$$\begin{aligned} \Pi_s^{new} &= \text{Poss}\{G = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \le a\} \\ &= \int_0^1 \frac{\max(0, w_{G^{\alpha}} - \max(0, G^{+\alpha} - a))}{w_{G^{\alpha}}} \mathrm{d}\alpha, \end{aligned} \quad (12)$$

$$\begin{aligned} \Pi_f^{new} &= \text{Poss}\{G = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) > a\} \\ &= \int_0^1 \frac{\max(0, w_{G^{\alpha}} - \max(0, a - G^{-\alpha}))}{w_{G^{\alpha}}} \mathrm{d}\alpha. \end{aligned} \quad (13)$$

- Case 2: the right hand side parameter ($\tilde{a}$) is a fuzzy number, and the left hand side parameter ($\boldsymbol{p}$) is a real number. The safety level between $[\tilde{a}^{-\alpha}, \tilde{a}^{+\alpha}]$ depends on the membership level of $\tilde{a}$. The possibilistic safety index and possibilistic failure index are modified to:

$$\begin{aligned} \Pi_s^{new} &= \text{Poss}\{G = g(\boldsymbol{x}, \boldsymbol{p}) \le \tilde{a}\} \\ &= \int_0^1 \frac{\max(0, w_{\tilde{a}^{\alpha}} - \max(0, G - \tilde{a}^{-\alpha}))}{w_{\tilde{a}^{\alpha}}} \mathrm{d}\alpha, \end{aligned} \quad (14)$$

$$\begin{aligned} \Pi_f^{new} &= \text{Poss}\{G = g(\boldsymbol{x}, \boldsymbol{p}) > \tilde{a}^{-\alpha}\} \\ &= \int_0^1 \frac{\max(0, w_{\tilde{a}^{\alpha}} - \max(0, a^{+\alpha} - G))}{w_{\tilde{a}^{\alpha}}} \mathrm{d}\alpha. \end{aligned} \quad (15)$$

- Case 3: both sides parameters ($\tilde{\boldsymbol{p}}$ and $\tilde{a}$) are fuzzy numbers, and the possibilistic safety index and possibilistic failure index are modified to:

$$\begin{aligned} \Pi_s^{new} &= \text{Poss}\{G = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \le \tilde{a}\} \\ &= \int_0^1 \frac{\max(0, w_{G^{\alpha}} + w_{\tilde{a}^{\alpha}} - \max(0, G^{+\alpha} - \tilde{a}^{-\alpha}))}{w_{G^{\alpha}} + w_{\tilde{a}^{\alpha}}} \mathrm{d}\alpha, \end{aligned} \quad (16)$$

$$\begin{aligned} \Pi_f^{new} &= \text{Poss}\{G = g(\boldsymbol{x}, \tilde{\boldsymbol{p}}) > \tilde{a}^{-\alpha}\} \\ &= \int_0^1 \frac{\max(0, w_{G^{\alpha}} + w_{\tilde{a}^{\alpha}} - \max(0, \tilde{a}^{+\alpha} - G^{-\alpha}))}{w_{G^{\alpha}} + w_{\tilde{a}^{\alpha}}} \mathrm{d}\alpha. \end{aligned} \quad (17)$$

Compared with the traditional reliability analysis in (8) and (9), the modified possibilistic safety index and possibilistic failure index can make use of the uncertain information more efficiently, which is more realistic in engineering practice [9].

Then, based on the analysis of the possibility that the constraint value (G) is greater than or less than the target value ($\tilde{a}$), in robust optimization process, we would like to select a solution with a smaller possibilistic failure index or larger possibilistic safety index, so that the possibility of satisfying the inequality constraints is higher. The fuzzy inequality constraints in Problem **P1** can be transformed to the following deterministic forms:

$$\max_{\boldsymbol{x} \in \mathbb{R}^n} \sum_{i=1}^{q} \Pi_{s,i}^{new}[g_i(\boldsymbol{x}, \tilde{\boldsymbol{p}})] \text{ or } \min_{\boldsymbol{x} \in \mathbb{R}^n} \sum_{i=1}^{q} \Pi_{f,i}^{new}[g_i(\boldsymbol{x}, \tilde{\boldsymbol{p}})]. \quad (18)$$

B. Feasibility robustness analysis for equality constraint

In order to improve the feasibility robustness, we want to find a solution whose equality constraint value is within the acceptable range and as close as possible to the ideal value. Cross-entropy can measure the degree of discrimination between two fuzzy variables. Therefore, we introduce the concept of fuzzy cross-entropy to evaluate the feasibility robustness of equality constraint. The definition of fuzzy cross-entropy [21][22] is as follows:

Definition 4. Let $\tilde{\beta}_1$ and $\tilde{\beta}_2$ denote fuzzy variables with assigned membership functions μ_1 and μ_2, respectively, then the cross-entropy of $\tilde{\beta}_1$ from $\tilde{\beta}_2$ is defined by

$$\begin{aligned} C(\tilde{\beta}_1, \tilde{\beta}_2) = &\int_{-\infty}^{+\infty} \mu_1(t) \log_2 \frac{\mu_1(t)}{\frac{1}{2}(\mu_1(t) + \mu_2(t))} \mathrm{d}t \\ &+ \int_{-\infty}^{+\infty} (1 - \mu_1(t)) \log_2 \frac{1 - \mu_1(t)}{1 - \frac{1}{2}(\mu_1(t) + \mu_2(t))} \mathrm{d}t. \end{aligned} \quad (19)$$

The cross-entropy $C(\tilde{\beta}_1, \tilde{\beta}_2)$ is not symmetric with respect to its arguments, and a symmetric cross-entropy can be constructed as:

$$I(\tilde{\beta}_1, \tilde{\beta}_2) = C(\tilde{\beta}_1, \tilde{\beta}_2) + C(\tilde{\beta}_2, \tilde{\beta}_1). \quad (20)$$

According to Shannon's inequality, it is easy to prove that $I(\tilde{\beta}_1, \tilde{\beta}_2) \ge 0$, where the equality holds if and only if $\tilde{\beta}_1 = \tilde{\beta}_2$.

Then, the discrimination between $H = h(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ and $\tilde{b}$ in Fig.2 (cases 4, 5, and 6) can be analyzed as:

- Case 4: the left hand side parameter ($\tilde{\boldsymbol{p}}$) is a fuzzy number, and the right hand side parameter (b) is a real number. To measure the distance between the constraint value $H = h(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ and the ideal value b, the discrimination index based on symmetric cross-entropy can be calculated as:

$$\Psi = I(H, b) = C(h(\boldsymbol{x}, \tilde{\boldsymbol{p}}), b) + C(b, h(\boldsymbol{x}, \tilde{\boldsymbol{p}})), \quad (21)$$

where the membership function of real number b can be considered as

$$\mu_b(t) = \begin{cases} 1 & \text{if } t = b \\ 0 & \text{if } t \ne b. \end{cases}$$

- Case 5: the right hand side parameter ($\tilde{b}$) is a fuzzy number, and the left hand side parameter ($\boldsymbol{p}$) is a real number. The ideal constraint value is $\tilde{b}^1$, and the satisfaction level between $[\tilde{b}^{-0}, \tilde{b}^{+0}]$ depends on the membership

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

level of $\tilde{b}$. We divide the fuzzy parameter $\tilde{b}$ into two parts: the lower bound set $\tilde{b}_l = [\tilde{b}^{-0}, \tilde{b}^1]$ and upper bound set $\tilde{b}_u = [\tilde{b}^1, \tilde{b}^{+0}]$. The discrimination index should consider not only the distance from ideal value, but also the satisfaction of the boundary, which can be calculated as:

$$\Psi = I(H, \tilde{b}^1) + \text{Poss}\{H < \tilde{b}_l\} + \text{Poss}\{H > \tilde{b}_u\}, \quad (22)$$

where the $\text{Poss}\{H < \tilde{b}_l\}$ and $\text{Poss}\{H > \tilde{b}_u\}$ can be obtained based on the analysis in Case 2.

- Case 6: both sides parameters ($\tilde{\boldsymbol{p}}$ and $\tilde{b}$) are fuzzy numbers, and the discrimination index can be calculated as:

$$\Psi = I(H, \tilde{b}^1) + \text{Poss}\{H < \tilde{b}_l\} + \text{Poss}\{H > \tilde{b}_u\}, \quad (23)$$

where the $\text{Poss}\{H < \tilde{b}_l\}$ and $\text{Poss}\{H > \tilde{b}_u\}$ can be obtained based on the analysis in Case 3.

From the above analysis about the closeness of the constraint value (H) to the target value ($\tilde{b}$), in robust optimization process, we will select a solution with a smaller discrimination index, so that the equality constraint value is not only closer to the ideal value but also with higher possibility to be within the acceptance range. Thus, the fuzzy equality constraints in Problem **P1** can be transformed to:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \quad \sum_{j=1}^{r} \Psi_j[h_j(\boldsymbol{x}, \tilde{\boldsymbol{p}})] \quad (24)$$

The robustness analyses for inequality constraints (18) and equality constraints (24) are of different dimensional orders. To emphasize both indices, a normalization procedure [2] is applied to transform the constraints in Problem **P1** to:

$$\begin{aligned}\min_{\boldsymbol{x}\in\mathbb{R}^n} \ (1-\omega) \sum_{i=1}^{q} \frac{\Pi_{f,i}^{new}[g_i(\boldsymbol{x}, \tilde{\boldsymbol{p}})]}{\max\limits_{i=1,\ldots,q} \{\Pi_{f,i}^{new}[g_i(\boldsymbol{x}, \tilde{\boldsymbol{p}})]\}} \\ + \omega \sum_{j=1}^{r} \frac{\Psi_j[h_j(\boldsymbol{x}, \tilde{\boldsymbol{p}})]}{\max\limits_{j=1,\ldots,r} \{\Psi_j[h_j(\boldsymbol{x}, \tilde{\boldsymbol{p}})]\}}\end{aligned} \quad (25)$$

where ω is a weighting factor; $\Pi_{f,i}^{new}$ measures the possibility of failure to satisfy an inequality constraint; Ψ_j is used to evaluate the discrimination between the equality constraint value and the ideal value. The value for ω is adjusted based on the complexity of the constraints in the optimization problem.

Based on the transformation models in (4), (5), and (25), the fuzzy robust optimization problem can be formulated as:

$$\textbf{Problem P2} \begin{cases} \min\limits_{\boldsymbol{x}\in\mathbb{R}^n} E[f(\boldsymbol{x}, \tilde{\boldsymbol{p}})] \\ \min\limits_{\boldsymbol{x}\in\mathbb{R}^n} D[f(\boldsymbol{x}, \tilde{\boldsymbol{p}})] \\ \min\limits_{\boldsymbol{x}\in\mathbb{R}^n} (1-\omega) \sum\limits_{i=1}^{q} \frac{\Pi_{f,i}^{new}[g_i(\boldsymbol{x}, \tilde{\boldsymbol{p}})]}{\max\limits_{i=1,\ldots,q} \{\Pi_{f,i}^{new}[g_i(\boldsymbol{x}, \tilde{\boldsymbol{p}})]\}} \\ \qquad + \omega \sum\limits_{j=1}^{r} \frac{\Psi_j[h_j(\boldsymbol{x}, \tilde{\boldsymbol{p}})]}{\max\limits_{j=1,\ldots,r} \{\Psi_j[h_j(\boldsymbol{x}, \tilde{\boldsymbol{p}})]\}}. \end{cases}$$

Problem **P2** is a multi-objective optimization problem, in which both optimality robustness and feasibility robustness are evaluated simultaneously. In most practical applications, the optimal solution for the constrained optimization problem is the candidate with the optimal objective function among the feasible solutions that satisfy the constraints. Therefore, in this paper, the feasibility robustness of a solution is more important than its optimality robustness. Considering the hierarchy between the feasibility robustness and the optimality robustness, we propose a game-theoretic optimization framework to rank the priorities of these three objective functions.

IV. Optimization Method Based on Stackelberg-Nash Game

Stackelberg-Nash game, as a noncooperative game strategy, involves one leader and several followers of equal status and is well suited for hierarchical decision-making modeling [15][23]. The leader (higher-level objective) could anticipate the followers' (lower-level objective) reaction and optimize its decision accordingly. After observing leader's play, the followers could optimize their objectives and reach a Nash equilibrium [24]. Considering that the objectives in Problem **P2** have different priorities, the optimization method based on the Stackelberg-Nash game is derived in this section.

A. Stackelberg-Nash game

Consider an $m+1$ player game with one leader and m followers. The payoff functions of leader and followers are represented as f_l and $f_{fi}(i = 1, \ldots, m)$, respectively. The search space of decision variable ($\boldsymbol{x} \in \mathbb{R}$) consists of the leader's search space ($\boldsymbol{x}_l \in \mathbb{R}_l$) and the followers' search space ($\boldsymbol{x}_{fi} \in \mathbb{R}_{fi}, i = 1, \ldots, m$). The multi-objective optimization problem based on the Stackelberg-Nash game can be defined by:

$$\begin{cases} \text{Leader:} & \min\limits_{\boldsymbol{x}_l \in\mathbb{R}_l} f_l(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \ldots, \boldsymbol{x}_{fm}) \\ \text{Follower 1:} & \min\limits_{\boldsymbol{x}_{f1}\in\mathbb{R}_{f1}} f_{f1}(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \ldots, \boldsymbol{x}_{fm}) \\ \cdots & \\ \text{Follower } m: & \min\limits_{\boldsymbol{x}_{fm}\in\mathbb{R}_{fm}} f_{fm}(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \ldots, \boldsymbol{x}_{fm}). \end{cases} \quad (26)$$

A Stackelberg-Nash equilibrium $\boldsymbol{x}^* = (\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}^*, \ldots, \boldsymbol{x}_{fm}^*)$ should satisfy the following conditions:

$$f_l(\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}^*, \ldots, \boldsymbol{x}_{fm}^*) = \inf_{\boldsymbol{x}_l\in\mathbb{R}_l} f_l(\boldsymbol{x}_l, \boldsymbol{x}_{f1}^*(\boldsymbol{x}_l), \ldots, \boldsymbol{x}_{fm}^*(\boldsymbol{x}_l)), \quad (27)$$

where $(\boldsymbol{x}_{f1}^*(\boldsymbol{x}_l), \ldots, \boldsymbol{x}_{fm}^*(\boldsymbol{x}_l))$ are the followers' reaction functions coming from the followers' Nash equilibrium:

$$\begin{cases} f_{f1}(\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}^*, \ldots, \boldsymbol{x}_{fm}^*) = \inf\limits_{\boldsymbol{x}_{f1}\in\mathbb{R}_{f1}} f_{f1}(\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}^*, \ldots, \boldsymbol{x}_{fm}^*) \\ \cdots \\ f_{fm}(\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}^*, \ldots, \boldsymbol{x}_{fm}^*) = \inf\limits_{\boldsymbol{x}_{fm}\in\mathbb{R}_{fm}} f_{fm}(\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}^*, \boldsymbol{x}_{f2}^*, \ldots, \boldsymbol{x}_{fm}). \end{cases} \quad (28)$$

In the context of the Nash equilibrium, it is assumed that each player knows the equilibrium strategies of others, and that one player cannot benefit from unilaterally changing its own strategy.

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

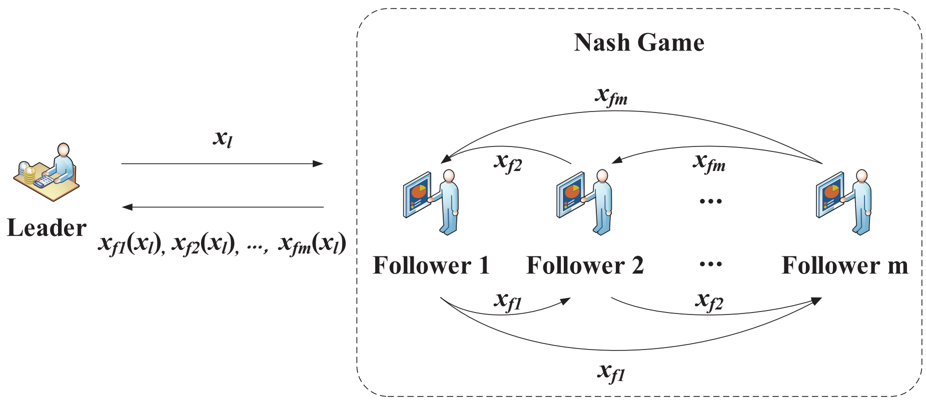


Fig. 3: Information flow of the Stackelberg-Nash game

Then, the leader and followers can find their optimal solution $\boldsymbol{x}^* = (\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}^*, \ldots, \boldsymbol{x}_{fm}^*)$ by solving

$$\frac{Df_l}{D\boldsymbol{x}_l}\bigg|_{\boldsymbol{x}^*} = \frac{\partial f_l}{\partial \boldsymbol{x}_l}\bigg|_{\boldsymbol{x}^*} + \sum_{i=1}^{m} \frac{\partial f_l}{\partial \boldsymbol{x}_{fi}} \frac{\partial \boldsymbol{x}_{fi}}{\partial \boldsymbol{x}_l}\bigg|_{(\boldsymbol{x}^*} = 0, \tag{29}$$

and

$$\frac{\partial f_{fi}}{\partial \boldsymbol{x}_{fi}}\bigg|_{\boldsymbol{x}^*} = 0, \quad i = 1, \ldots, m. \tag{30}$$

The information flow of the Stackelberg-Nash game is shown in Fig.3. The leader provides the decision information $\boldsymbol{x}_l$ to the followers. Each follower passes its decision to others and also receives the decision of other followers. After the followers reaching the Nash equilibrium, the leader will receive the reaction information from the followers. The reaction functions $(\boldsymbol{x}_{f1}^*(\boldsymbol{x}_l), \ldots, \boldsymbol{x}_{fm}^*(\boldsymbol{x}_l))$ can be computed or approximated by performing a sensitivity analysis [15] that introduces a small change in $\boldsymbol{x}_l$.

Based on the Stackelberg-Nash game strategy, Problem **P2** can be transformed to:

$$\textbf{Problem P3} \begin{cases} \text{Leader: } \min\limits_{\boldsymbol{x}_l \in \mathbb{R}^l} (1-\omega) \sum\limits_{i=1}^{q} \dfrac{\Pi_{f,i}^{new}[g_i(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})]}{\max\limits_{i=1,\ldots,q} \{\Pi_{f,i}^{new}[g_i(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})]\}} \\ \qquad + \omega \sum\limits_{j=1}^{r} \dfrac{\Psi_j[h_j(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})]}{\max\limits_{j=1,\ldots,r} \{\Psi_j[h_j(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})]\}} \\ \text{Follower 1: } \min\limits_{\boldsymbol{x}_{f1} \in \mathbb{R}^{f1}} E[f(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})] \\ \text{Follower 2: } \min\limits_{\boldsymbol{x}_{f2} \in \mathbb{R}^{f2}} D[f(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})]. \end{cases}$$

B. Leader-followers state transition algorithm

The state transition algorithm (STA) [25][26][27] is a global optimization method. It consists of four stochastic search operators. The main idea behind the STA method is the control theory of state transition and state space representation.

Given a solution $\boldsymbol{x}^k$, the candidates can be generated by the following four operators: Rotation transformation:

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \eta_1 \frac{1}{\|\boldsymbol{x}^k\|_2} R_1 \boldsymbol{x}^k, \tag{31}$$

Translation transformation:

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \eta_2 R_2 \frac{\boldsymbol{x}^k - \boldsymbol{x}^{k-1}}{\|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|_2}, \tag{32}$$

Expansion transformation:

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \eta_3 R_3 \boldsymbol{x}^k, \tag{33}$$

Axesion transformation:

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \eta_4 R_4 \boldsymbol{x}^k, \tag{34}$$

where k is the number of iterations; parameters η_1, η_2, η_3, and η_4 are rotation factor, translation factor, expansion factor, and axesion factor, respectively, which can control the size of search range; the matrixes R_1, R_2, R_3, and R_4 have special elements that can control the shape of search range. More specifically, the elements of R_1 are uniformly distributed random variables defined in the interval $[-1, 1]$; the elements of R_2 are uniformly distributed random variable defined in the interval $[0, 1]$; R_3 is a random diagonal matrix with its elements obeying the Gaussian distribution; and R_4 is a random diagonal matrix with its elements obeying the Gaussian distribution, and only one random position has nonzero value [25].

To solve the optimization problem in Problem **P3**, we design a leader-followers state transition algorithm. First, the state transition algorithm runs sequentially on different followers to find a Nash equilibrium, which is shown in Algorithm 1. In order to converge to a Nash equilibrium, a sufficient number of iterations $Iter_f$ is required.

Algorithm 1 Followers-based state transition algorithm

Input:
$Iter_f$: maximum number of iterations for followers' optimization
$(\boldsymbol{x}_l^0, \boldsymbol{x}_{f1}^0, \boldsymbol{x}_{f2}^0)$: initial solution
Output:
$(\boldsymbol{x}_{f1}^*, \boldsymbol{x}_{f2}^*)$: Nash equilibrium
for k = 1 **to** $Iter_f$ **do**
  Generating candidate solutions for followers using state transition operators in (31) to (34)
  Updating the decision $\boldsymbol{x}_{f1}^k$ of follower 1 $f_{f1}(\boldsymbol{x}_l^0, \boldsymbol{x}_{f1}^k, \boldsymbol{x}_{f2}^0)$ by STA

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

Updating the decision $\boldsymbol{x}_{f2}^k$ of follower 2 $f_{f2}(\boldsymbol{x}_l^0, \boldsymbol{x}_{f1}^0, \boldsymbol{x}_{f2}^k)$ by STA
$\boldsymbol{x}_{f1}^0 \leftarrow \boldsymbol{x}_{f1}^k$
$\boldsymbol{x}_{f2}^0 \leftarrow \boldsymbol{x}_{f2}^k$
end for
$(\boldsymbol{x}_{f1}^*, \boldsymbol{x}_{f2}^*) \leftarrow (\boldsymbol{x}_{f1}^{Iter_f}, \boldsymbol{x}_{f2}^{Iter_f})$

Then, the leader's optimization procedure is shown in Algorithm 2. The reaction information $\frac{\partial x_{f1}}{\partial \boldsymbol{x}_l}$ and $\frac{\partial x_{f2}}{\partial \boldsymbol{x}_l}$ can be computed by perturbing x_l and repeating the followers optimization.

Algorithm 2 Leader-based state transition algorithm

Input:
$Iter_l$: maximum number of iterations for leader's optimization
$(\boldsymbol{x}_l^0, \boldsymbol{x}_{f1}^0, \boldsymbol{x}_{f2}^0)$: initial solution
Output:
$(\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}^*, \boldsymbol{x}_{f2}^*)$: Stackelberg-Nash equilibrium
for k = 1 **to** $Iter_l$ **do**
Obtaining the reaction information $\frac{\partial x_{f1}}{\partial \boldsymbol{x}_l}, \frac{\partial x_{f2}}{\partial \boldsymbol{x}_l}$ from Algorithm 1
Generating candidate solutions for leader using state transition operators (31) to (34)
$\boldsymbol{x}_{f1}^k \leftarrow \boldsymbol{x}_{f1}^{k-1} + \frac{\partial x_{f1}}{\partial \boldsymbol{x}_l}(\boldsymbol{x}_l^k - \boldsymbol{x}_l^{k-1})$
$\boldsymbol{x}_{f2}^k \leftarrow \boldsymbol{x}_{f2}^{k-1} + \frac{\partial x_{f2}}{\partial \boldsymbol{x}_l}(\boldsymbol{x}_l^k - \boldsymbol{x}_l^{k-1})$
Updating the decision $\boldsymbol{x}_l^k$ of leader $f_l(\boldsymbol{x}_l^k, \boldsymbol{x}_{f1}^k, \boldsymbol{x}_{f2}^k)$ by STA
end for
$(\boldsymbol{x}_l^*, \boldsymbol{x}_{f1}^*, \boldsymbol{x}_{f2}^*) \leftarrow (\boldsymbol{x}_l^{Iter_l}, \boldsymbol{x}_{f1}^{Iter_l}, \boldsymbol{x}_{f2}^{Iter_l})$

Finally, with the leader and followers' optimization procedures, the robust optimization problem is solved hierarchically and the Stackelberg-Nash equilibrium is obtained achieve better feasibility robustness.

C. Optimization framework

The framework of the fuzzy robust optimization method based on the Stackelberg-Nash game (with one leader and two followers) is shown in Fig.4. First, in order to quantitatively analyze the optimality robustness and feasibility robustness of Problem **P1**, the expectation-entropy model is established to evaluate the performance of fuzzy objective function, and a comprehensive feasibility robustness analysis method is proposed to evaluate the performance of fuzzy constraints. Then, the optimization problem with fuzzy parameters is transformed to a multi-objective optimization problem (Problem **P2**). Considering that the feasibility robustness has higher priority than the optimality robustness, the Stackelberg-Nash game is introduced to establish an optimization problem with one leader and two followers (Problem **P3**). Finally, the leader-followers state transition algorithm is conducted to search for a Stackelberg-Nash equilibrium.

V. Examples and Results

In this section, the effectiveness of the fuzzy robust optimization method based on the Stackelberg-Nash game is verified via two practical engineering applications: (i) optimization for crashworthiness of vehicle side impact, and (ii) optimization for power scheduling. The optimization framework shown in Fig.4 is used to obtain the equilibrium solution considering both feasibility robustness and optimality robustness.

In both examples, the parameter settings of state transition algorithm are the same as in [25][26][27]. The algorithm can solve the optimization problems for such applications as benchmarks, image segmentation, and process control. The transformation factors are: $\eta_1 = 1, \eta_2 = 1, \eta_3 = 1, \eta_4 = 1$. The search enforcement (SE) is 30 and iteration number is 15. The maximum numbers of iteration for followers' optimization and leader's optimization are both set to 20. The decision variables for the leader and followers are assigned based on their spatial distances [28]. All results are obtained by MATLAB R2016a software.

A. Case 1: Optimization for crashworthiness of vehicle side impact

The crashworthiness of vehicle side impact is important in the structure evaluation system [29]. The existence of uncertainties in manufacturing process calls for a reliable and robust design. Therefore, we study the optimization of vehicle side impact with fuzzy uncertainties, aiming to illustrate the effectiveness of the proposed robust optimization method.

The vehicle side impact model (shown in Fig.5) is designed to minimize the vehicle weight ($f(\cdot)$) as well as meet internal and regulated side impact constraints ($g(\cdot)$) specified by the vehicle safety requirements [19]:

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) = 1.98 + 4.90x_1 + 6.67x_2 + 6.98x_3 + 4.01x_4 \\
& \qquad + 1.78x_5 + 2.73x_7 \\
\text{s.t.} \quad & g_1(\boldsymbol{x}, \boldsymbol{p}) = 1.16 - 0.3717x_2x_4 - 0.00931x_2p_1 \\
& \qquad - 0.484x_3x_9 + 0.01343x_6p_1 \leq 1 \\
& g_2(\boldsymbol{x}, \boldsymbol{p}) = 28.98 + 3.818x_3 - 4.2x_1x_2 + 0.0207x_5p_1 \\
& \qquad + 6.63x_6x_9 - 7.7x_7x_8 + 0.32x_9p_1 \leq 32 \\
& g_3(\boldsymbol{x}, \boldsymbol{p}) = 33.86 + 2.95x_3 + 0.1792p_1 - 5.057x_1x_2 \\
& \qquad - 11x_2x_8 - 0.0215x_5p_1 - 9.98x_7x_8 \\
& \qquad + 22x_8x_9 \leq 32 \\
& g_4(\boldsymbol{x}, \boldsymbol{p}) = 46.36 - 9.9x_2 - 12.9x_1x_8 \\
& \qquad + 0.1107x_3p_1 \leq 32 \\
& g_5(\boldsymbol{x}, \boldsymbol{p}) = 0.261 - 0.0159x_1x_2 + 0.0008757x_5p_1 \\
& \qquad - 0.019x_2x_7 + 0.0144x_3x_5 - 0.188x_1x_8 \\
& \qquad + 0.08045x_6x_9 + 0.00139x_8p_2 \\
& \qquad + 0.00001575p_1p_2 \leq 0.32 \qquad\qquad (35) \\
& g_6(\boldsymbol{x}, \boldsymbol{p}) = 0.214 + 0.00817x_5 - 0.131x_1x_8 \\
& \qquad - 0.0704x_1x_9 + 0.0007715x_5p_1 - 0.018x_2x_7 \\
& \qquad + 0.0208x_3x_8 + 0.121x_3x_9 - 0.00364x_5x_6 \\
& \qquad - 0.02x_2^2 - 0.0005354x_6p_1 + 0.00121x_8p_2 \\
& \qquad + 0.00184x_9p_1 + 0.03099x_2x_6 \leq 0.32 \\
& g_7(\boldsymbol{x}, \boldsymbol{p}) = 0.74 - 0.61x_2 - 0.163x_3x_8 + 0.001232x_3p_1 \\
& \qquad - 0.166x_7x_9 + 0.227x_2^2 \leq 0.32 \\
& g_8(\boldsymbol{x}, \boldsymbol{p}) = 4.72 - 0.5x_4 - 0.19x_2x_3 - 0.0122x_4p_1
\end{aligned}
$$

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

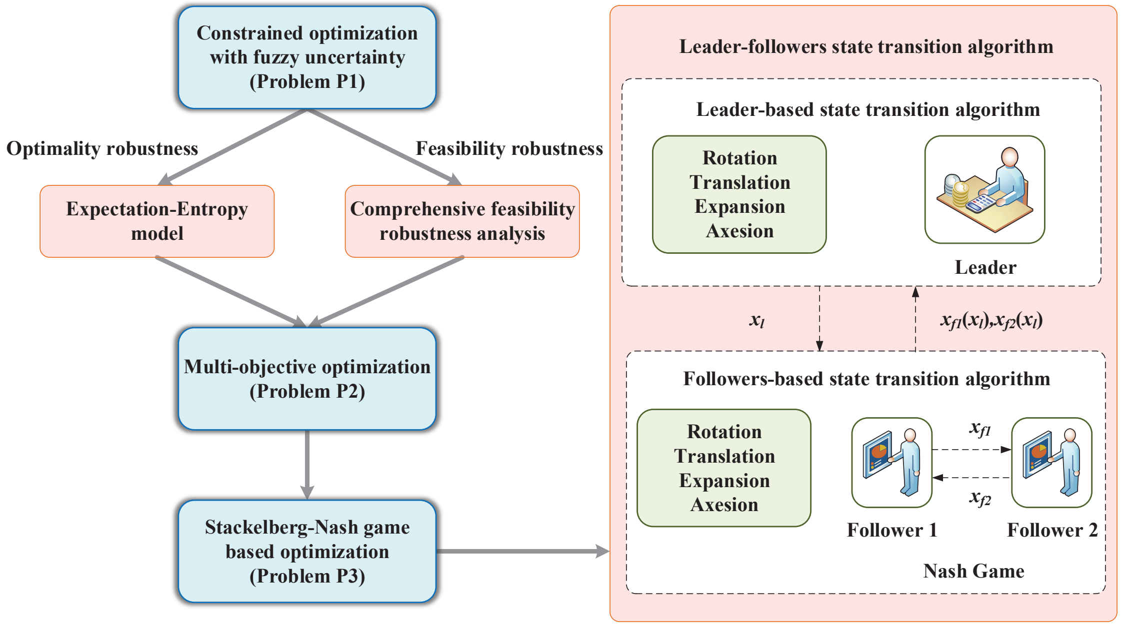


Fig. 4: Framework of fuzzy robust optimization based on the Stackelberg-Nash game

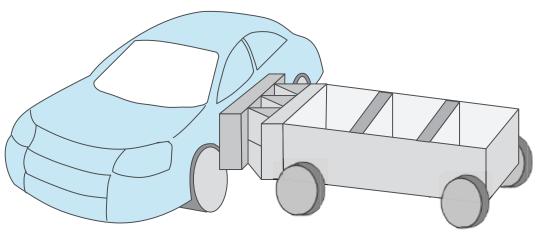


Fig. 5: Vehicle side impact model [30]

$$
\begin{aligned}
&+0.009325x_6p_1+0.000191p_2^2\leq 4\\
g_9(\boldsymbol{x},\boldsymbol{p}) &= 10.58-0.0674x_1x_2+0.02054x_3p_1\\
&-1.95x_2x_8-0.0198x_4p_1+0.028x_6p_1\leq 9.9\\
g_{10}(\boldsymbol{x},\boldsymbol{p}) &= 16.45-0.843x5x_6+0.0432x_9p_1\\
&-0.489x_3x_7-0.0556x_9p_2-0.000786p_2^2\leq 15.7\\
&0.5\leq x_n\leq 1.5\ (n=1,2,\ldots,7)\\
&x_n=0.192 \text{ or } 0.345\ (n=8,9)\\
&\boldsymbol{p}=[0,0]
\end{aligned}
$$

where the decision variables are the thickness ($x_1,\ldots,x_7$) and material properties of critical parts (x_8,x_9); the parameters p_1 and p_2 represent the barrier height and hitting position; the constraints are the abdomen load (g_1), lower rib deflection (g_2), middle rib deflection (g_3), upper rib deflection (g_4), lower rib viscous criterion (g_5), middle rib viscous criterion (g_6), upper rib viscous criterion (g_7), public symphysis force (g_8), velocity of B-pillar at middle point (g_9), and velocity of front door at B-pillar (g_{10}). Taking into account the uncertainty in the manufacturing process [31], the uncertain design variables and parameters are modeled with triangular fuzzy variables:

$$
\tilde{x_n}=(x_n-0.09,x_n,x_n+0.09),\ \ n=1,2,3,4,6,7
$$

$$
\tilde{x_5}=(x_5-0.15,x_5,x_5+0.15),\ \ \tilde{p_1}=(-30,0,30),\\
\tilde{p_2}=(-30,0,30).
$$

The objective function of the vehicle side impact model is a linear function. According to the fuzzy uncertainty theory, the fuzzy entropy of $f(\cdot)$ is a fixed number, and there is no need to consider the fuzzy entropy in the formulation of robust optimization problem. Thus, we only consider the expectation of fuzzy objective function and the possibilistic failure index of fuzzy constraints. Based on the robustness analysis and the Stackelberg-Nash game strategy, the fuzzy robust optimization model for vehicle side impact is represented as:

$$
\begin{cases}
\text{Leader:} & \min\limits_{\boldsymbol{x}_l} \ \sum\limits_{i=1}^{10}\Pi_{f,i}^{new}[g_i(\tilde{\boldsymbol{x}}_l,\tilde{\boldsymbol{x}_f},\tilde{\boldsymbol{p}})]\\
\text{Follower:} & \min\limits_{\boldsymbol{x}_f} \ E[f(\tilde{\boldsymbol{x}}_l,\tilde{\boldsymbol{x}_f},\tilde{\boldsymbol{p}})].
\end{cases}
\tag{36}
$$

To verify the effectiveness of the feasibility robustness analysis method, we compute the possibilistic failure indexes of two candidates based on the traditional reliability model and the modified model. For two solutions $\boldsymbol{x}(1)$ = (0.5,1,1,1,0.5,1.5,0.5,0.3450,0.3450) and $\boldsymbol{x}(2)$ = (0.5,1,1.2,1,0.5,1.5,0.5,0.3450,0.3450), the feasibility robustness for constraint g_1 is analyzed in Table.I.

It can be observed that the possibilistic failure indexes of $\boldsymbol{x}(1)$ and $\boldsymbol{x}(2)$ are the same under the traditional reliability model. Once the upper bound of the interval abdomen load exceeds the critical load, it can be considered that the constraint of g_1 fails. Thus, the traditional reliability model is too conservative for analyzing the feasibility robustness accurately. With the modified reliability model, the possibilistic failure indexes of $\boldsymbol{x}(1)$ and $\boldsymbol{x}(2)$ have different values, enabling the information in the entire uncertain space to be analyzed. Therefore, the modified reliability model is more realistic for analyzing the feasibility robustness.

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

TABLE I: Analysis of feasibility robustness for constraint g_1

Feasibility robustness	Membership level α	Traditional model $\boldsymbol{x}(1)$	Traditional model $\boldsymbol{x}(2)$	Modified model $\boldsymbol{x}(1)$	Modified model $\boldsymbol{x}(2)$
Interval possibilistic failure indexes	0.0	1.0000	1.0000	0.1047	0.0694
	0.1	1.0000	1.0000	0.0522	0.0127
	0.2	0.0000	0.0000	0.0000	0.0000
	0.3	0.0000	0.0000	0.0000	0.0000
	0.4	0.0000	0.0000	0.0000	0.0000
	0.5	0.0000	0.0000	0.0000	0.0000
	0.6	0.0000	0.0000	0.0000	0.0000
	0.7	0.0000	0.0000	0.0000	0.0000
	0.8	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000
	1.0	0.0000	0.0000	0.0000	0.0000
Fuzzy possibilistic failure indexes		0.1500	0.1500	0.0105	0.0047

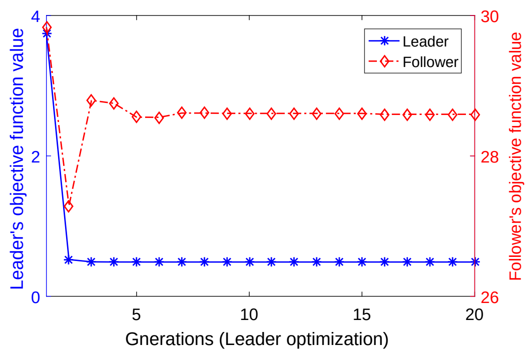


Fig. 6: Convergent trajectories of the optimization using leader-followers state transition algorithm

The convergent trajectories of the optimization using leader-followers state transition algorithm are shown in Fig.6. It can be observed that both the leader and follower can converge to stable values after several iterations. The leader can achieve higher satisfaction than its follower. Thus, the leader-followers state transition algorithm is able to efficiently find the optimal solution with higher feasibility robustness.

TABLE II: Optimal solutions for vehicle side impact problem

	Deterministic design	Reliability-based design [31]	Proposed robust design
(x_1, x_2, x_3)	$(0.5000, 1.2257, 0.5000)$	$(1.5000, 1.1640, 1.0930)$	$(0.5900, 1.2600, 0.5900)$
(x_4, x_5, x_6)	$(1.2071, 0.8752, 1.1250)$	$(1.2310, 2.2220, 0.6450)$	$(1.4100, 2.3638, 0.5779)$
(x_7, x_8, x_9)	$(0.4000, 0.3450, 0.1920)$	$(0.9440, 0.3450, 0.3450)$	$(0.4900, 0.3450, 0.1920)$
$(\Pi^{new}_{f,1}, \Pi^{new}_{f,2})$	$(0.0000, 0.0107)$	$(0.0000, 0.0000)$	$(0.0000, 0.0041)$
$(\Pi^{new}_{f,3}, \Pi^{new}_{f,4})$	$(0.0398, 0.4984)$	$(0.0000, 0.0253)$	$(0.0058, 0.2299)$
$(\Pi^{new}_{f,5}, \Pi^{new}_{f,6})$	$(0.0000, 0.0000)$	$(0.0000, 0.0000)$	$(0.0105, 0.0000)$
$(\Pi^{new}_{f,7}, \Pi^{new}_{f,8})$	$(0.0000, 0.5471)$	$(0.0000, 0.1810)$	$(0.0000, 0.2213)$
$(\Pi^{new}_{f,9}, \Pi^{new}_{f,10})$	$(0.2026, 0.0967)$	$(0.1624, 0.0000)$	$(0.0155, 0.0013)$
$E[f]$	23.5860	36.1916	28.5927
$\sum \Pi^{new}_{f,i}[g_i]$	1.3954	0.3687	0.4883

The comparison optimization results of deterministic design, reliability-based design, and the robust design based on the Stackelberg-Nash game are shown in Table.II. The deterministic design applies the standard state transition algorithm with the same parameter settings in [25] to search for the optimal solution. As a traditional method to optimize the crashworthiness of vehicle side impact, the deterministic design takes no account of the fuzzy uncertainties in variables. The reliability-based design in [31] has strict requirements on feasibility robustness by ensuring the achievement of the constraints at the desired level, but it dose not consider the robustness performance of objective functions. From Table.II, it can be observed that the feasibility robustness of the deterministic solution is weak. The solution obtained by the reliability-based design has a lower failure possibility and satisfies the requirements of 7 constraints. Note that the optimal solution of reliability-based design is too conservative, leading to an increase in the vehicle weight, which is uneconomic for practical applications. The optimization performance in Table.II shows that the robust optimization based on the Stackelberg-Nash game can provide a solution not only with high feasibility robustness but also with reasonable vehicle weight.

To analyze the computational performance of the proposed robust optimization method, a total of 20 independent runs are conducted. The average execution time of proposed robust design is 365.54 seconds, which is much larger than that of the deterministic design (5.88 seconds). This is due to the high computational cost of robustness analysis, including a nested optimization structure. In the deterministic design, each candidate only needs to calculate the values of its objective function and constraints; while in the proposed robust design, each candidate needs to evaluate its performance under fuzzy uncertainties, including sensitive analysis at several membership levels.

B. Case 2: Optimization for power scheduling in zinc electrowinning process

Optimal scheduling of power usage is important in the industrial process. The zinc electrowinning process accounts for 80% of the total energy consumption of zinc hydrometallurgy. Under the power time-of-use pricing policy, the production operation in different periods should be adjusted according to the electricity price [32]. Due to the incomplete knowledge of the process model, some parameters in the power scheduling optimization problem are uncertain. Therefore, we study the power scheduling optimization problem to verify the feasibility of the proposed robust optimization method in complex industrial processes.

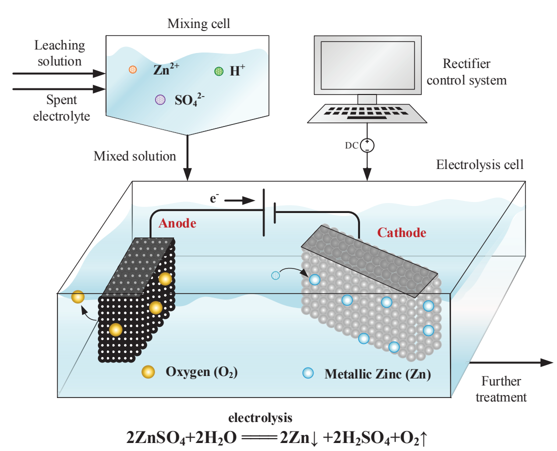


Fig. 7: Electrolytic cell of zinc electrowinning process [32]

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

A zinc electrowinning process includes several series potrooms and each potroom has several parallel electrolytic cells (shown in Fig.7). In order to minimize the electricity charge ($f(\cdot)$) and satisfy the requirement of zinc daily output ($h(\cdot)$), the optimization model of power scheduling in zinc electrowinning process is formulated as:

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & f(\boldsymbol{x},\boldsymbol{p}) = J_0 + \sum_{i=1}^{3}\sum_{j=1}^{7} P_i T_i V_{ij}(\boldsymbol{x},\boldsymbol{p}) L_{ij}(\boldsymbol{x}) \\
\text{s.t.} \quad & h(\boldsymbol{x},\boldsymbol{p}) = \sum_{i=1}^{3}\sum_{j=1}^{7} dR_i(\boldsymbol{x},\boldsymbol{p}) T_i L_{ij}(\boldsymbol{x}) = b \\
& 100 \le \boldsymbol{x}_1 \le 650,\ 45 \le \boldsymbol{x}_2 \le 60,\ 160 \le \boldsymbol{x}_3 \le 200 \\
\text{where} \quad & V_{ij}(\boldsymbol{x},\boldsymbol{p}) = N_j(p_1 - p_2 \ln(p_3 x_{3,i}^{-1}) - p_4 \ln(p_5 x_{2,i}) \\
& \qquad + p_6 x_{1,i}(p_7 + p_8 x_{3,i} - p_9 x_{2,i})^{-1} \\
& \qquad + p_{10} \lg x_{1,i} + p_{11} x_{1,i}) \\
& L_{ij}(\boldsymbol{x}) = B_j s x_{1,i} \\
& R_i(\boldsymbol{x},\boldsymbol{p}) = p_{12} \exp(p_{13} + p_{14} \lg x_{1,i}) x_{2,i}^{1.6} x_{3,i}^{-0.2} \\
& \qquad + (p_{15} \exp(-p_{16} + p_{17} \lg x_{1,i}) x_{2,i} x_{3,i}^{-0.2} \\
& \qquad + p_{18} + p_{19} x_{2,i}^{0.6})^{-1} x_{1,i}^{-1} \qquad (37) \\
& d = 1.2202\text{g/Ah}, \quad s = 1.13\text{m}^2, \quad b = 960\text{tons} \\
& \boldsymbol{T} = [11, 5, 8]\text{h} \\
& \boldsymbol{P} = 0.5627 \times [1.6, 1.0, 0.7]\text{¥/kWh} \\
& \boldsymbol{N} = [240, 240, 246, 192, 208, 208, 208] \\
& \boldsymbol{B} = [34, 46, 54, 56, 56, 57, 57] \\
& \boldsymbol{p} = [1.588, 0.027, 1.1025 \times 10^{-12}, 0.0135, 8.15, \\
& \qquad 6.2 \times 10^{-4}, 0.5931, 0.0181, 0.0313, \\
& \qquad 0.0793, 5 \times 10^{-4}, 1091.46, -4.06, \\
& \qquad 2.8, 0.0813, 1.8, 3.5, 0.35, 2172.45],
\end{aligned}
$$

where the decision variables are the current density ($x_{1,i}$), the concentration of Zn^{2+} ($x_{2,i}$), and the concentration of H^+ ($x_{3,i}$) in the ith period; parameters ($\boldsymbol{p}$) are the model coefficients based on the electrochemical reaction mechanism and historical data; J_0 is the capacity electricity charge with $J_0 = k_c \cdot \varphi$, where k_c is a fixed electricity price and φ is the maximum capacity of the transformers; P_i means the electricity price of the ith period; T_i is the duration of the ith period; V_{ij} is the cell voltage of the jth plant in the ith period; L_{ij} is the current magnitude of the electrolysis process of the jth plant during the ith period; b is the ideal zinc daily output; d is the electrochemical equivalent of zinc; R_i is the current-efficiency during the ith period; N_j and B_j are the numbers of cells and plates in a cell of the jth plant; and s is the area of the cathode plate. To account for the uncertainties in process modeling, some of the parameters are modeled with triangular fuzzy variables:

$$
\begin{aligned}
\tilde{p_n} &= (0.9 \times p_n, p_n, 1.1 \times p_n), n = 3, 4, 6, \ldots, 11 \\
\tilde{p_n} &= (0.99 \times p_n, p_n, 1.01 \times p_n), n = 12, 17 \\
\tilde{p_n} &= (0.98 \times p_n, p_n, 1.02 \times p_n), n = 13, 16, 19 \\
\tilde{b} &= (945, 960, 975).
\end{aligned}
$$

Using the robustness analysis and the Stackelberg-Nash game strategy, the fuzzy robust optimization model for power scheduling is formulated as:

$$
\begin{cases}
\text{Leader:} & \min_{\boldsymbol{x}_l} \ \Psi[h(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})] \\
\text{Follower 1:} & \min_{\boldsymbol{x}_{f1}} \ E[f(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})] \\
\text{Follower 2:} & \min_{\boldsymbol{x}_{f2}} \ D[f(\boldsymbol{x}_l, \boldsymbol{x}_{f1}, \boldsymbol{x}_{f2}, \tilde{\boldsymbol{p}})].
\end{cases} \tag{38}
$$

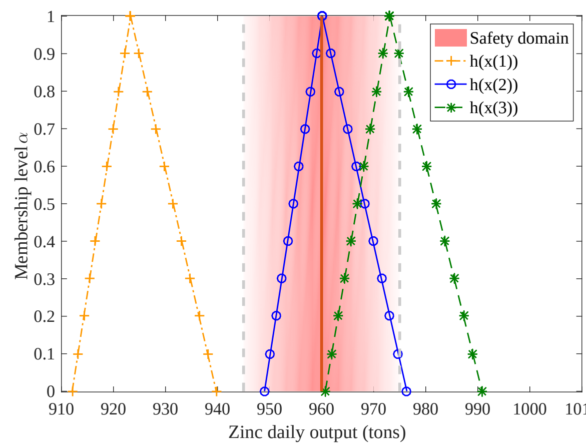


Fig. 8: Constraint values with candidate solutions $\boldsymbol{x}(1), \boldsymbol{x}(2)$, and $\boldsymbol{x}(3)$

TABLE III: Analysis of feasibility robustness for constraint h

Candidate solutions	$\boldsymbol{x}(1)$	$\boldsymbol{x}(2)$	$\boldsymbol{x}(3)$
Discrimination indexes	41.2126	19.1023	23.2362

To evaluate the feasibility robustness of candidate solution for equality constraint in the power scheduling optimization problem, the discrimination between constraint value and ideal value is analyzed. Given three candidate solutions $\boldsymbol{x}(1) = (160, 500, 600, 60, 60, 60, 200, 200, 200)$, $\boldsymbol{x}(2) = (421, 484, 462, 45, 50, 45, 200, 199, 161)$, and $\boldsymbol{x}(3) = (100, 650, 650, 60, 60, 60, 200, 200, 200)$, their constraint values are fuzzy variables which are shown in Fig.8, and their discrimination indexes are listed in Table.III. It is obvious that the zinc daily output under candidate $\boldsymbol{x}(1)$ is beyond the satisfactory range. The zinc daily output under candidate $\boldsymbol{x}(2)$ is close to the ideal value, and the discrimination indexes between $h(\boldsymbol{x}(2))$ and b is the smallest. Therefore, the proposed feasibility robustness analysis method is effective for evaluating the uncertainties of fuzzy equality constraint.

The convergent trajectories of the leader-followers state transition algorithm are shown in Fig.9. The optimization method based on the Stackelberg-Nash game strategy can converge to a solution with higher satisfaction of feasibility robustness (leader) and good satisfaction of optimality robustness (follower 1 and follower 2).

The results of the deterministic optimization, the reliability-based optimization, and the proposed robust optimization are shown in Table.IV. The fuzzy variables of $f(\boldsymbol{x})$ and $h(\boldsymbol{x})$

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

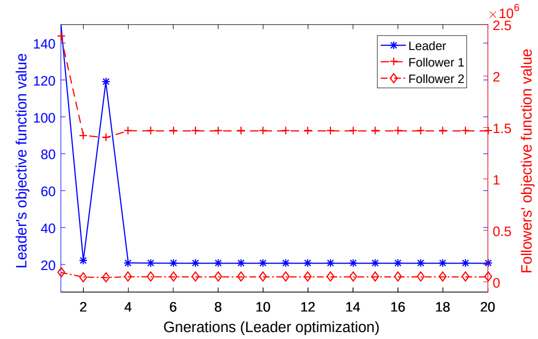


Fig. 9: Convergent trajectories of the optimization using leader-followers state transition algorithm

TABLE IV: Optimal solutions for power scheduling in zinc electrowinning process

	Deterministic design	Reliability-based design	Proposed robust design
$(x_{1,1}, x_{1,2}, x_{1,3})$	$(117, 638, 598)$	$(334, 226, 650)$	$(100, 650, 650)$
$(x_{2,1}, x_{2,2}, x_{2,3})$	$(60, 60, 60)$	$(60, 60, 45)$	$(60, 56, 60)$
$(x_{3,1}, x_{3,2}, x_{3,3})$	$(190, 176, 162)$	$(160, 200, 168)$	$(197, 200, 200)$
$E[f]$	1.4635×10^6	1.7359×10^6	1.4670×10^6
$D[f]$	5.7161×10^4	5.8382×10^4	4.7754×10^4
$\Psi[h]$	20.6371	18.9673	20.6256
Time(second)		1055.0934	1116.0213

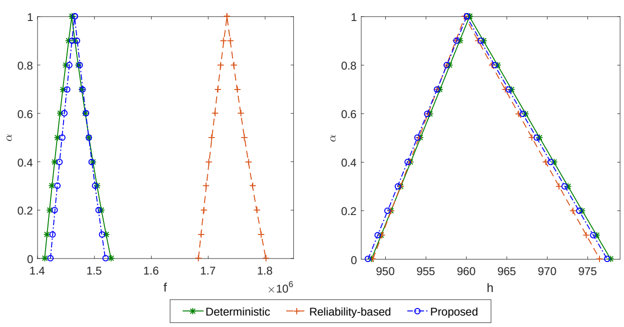


Fig. 10: Fuzzy variables of $f(\boldsymbol{x})$ and $h(\boldsymbol{x})$ under the three optimization methods

under the three optimization methods are shown in Fig.10. The deterministic optimization method dose not consider the parameter uncertainties, and its optimality robustness ($D[f]$) is weak. In reliability-based design, the desired level of the achievement for the constraint is set to be no worse than 97% of the optimal constraint performance (only consider the discrimination index in (38)). With the strict requirement on feasibility robustness, the reliability-base design achieves the smallest value of the discrimination index. Fig.10 also shows that the zinc daily output optimized by the reliability-based design is the closest to the ideal value (960tons), but its electricity charge is much higher than other methods. Base on the proposed robust optimization method, the possible constraint value has smaller discrimination index, and the function $f(\boldsymbol{x})$ gives better average performance and smaller variations.

The average execution times of the deterministic design, the reliability-based design, and the proposed robust design are 9.72 seconds, 752.02 seconds, and 1116.02 seconds, respectively. In the deterministic design, the candidate solution dose not consider the impact of uncertainties. In the reliability-based design, each candidate needs to calculate its objective function value and analyze the performance of constraints under fuzzy uncertainties. In the proposed robust design, each candidate needs to evaluate the performance of both objective and constraints under fuzzy uncertainties. Therefore, the computational cost of the proposed robust design is higher than that of other methods.

VI. Conclusion

This paper presented a new robust optimization method to solve the constrained optimization problem with fuzzy variables. A fuzzy expectation-entropy model is established to evaluate the optimality robustness, while a comprehensive analysis method is proposed to assess the feasibility robustness. An optimization framework with the Stackelberg-Nash game strategy is developed to analyze the hierarchical relationship in objectives. The state transition algorithm is given to optimize the objectives for leader and followers sequentially and obtain a equilibrium solution. Practical examples on vehicle side impact design and power scheduling design are used to verify the effectiveness of the proposed new design techniques. The experimental results showed that the obtained solution not only has higher satisfaction rate on feasibility robustness, but also can guarantee the optimality robustness. In the future, we will develop effective robustness analysis methods to reduce computational costs, and consider the robust optimization problem with complicated uncertainties.

References

[1] B. I. Yatsalo and L. Martínez, "Fuzzy rank acceptability analysis: A confidence measure of ranking fuzzy numbers," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 6, pp. 3579–3593, 2018.
[2] S. Bhattacharjya and S. Chakraborty, "Robust optimization of structures subjected to stochastic earthquake with limited information on system parameter uncertainty," *Engineering Optimization*, vol. 43, no. 12, pp. 1311–1330, 2011.
[3] J. Cheng, Z. Liu, Y. Qian, D. Wu, Z. Zhou, W. Gao, J. Zhang, and J. Tan, "Robust optimization of uncertain structures based on interval closeness coefficients and the 3D violation vectors of interval constraints," *Structural and Multidisciplinary Optimization*, vol. 60, no. 1, pp. 17–33, 2019.
[4] M. S. Pishvaee and M. F. Khalaf, "Novel robust fuzzy mathematical programming methods," *Applied Mathematical Modelling*, vol. 40, no. 1, pp. 407–418, 2016.
[5] M. S. Pishvaee, J. Razmi, and S. A. Torabi, "Robust possibilistic programming for socially responsible supply chain network design: A new approach," *Fuzzy Sets and Systems*, vol. 206, pp. 1–20, 2012.
[6] X. Huang, "Mean-entropy models for fuzzy portfolio selection," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 4, pp. 1096–1101, 2008.
[7] B. Liu, *Uncertainty theory*. Springer, 2010.
[8] C. Fan, Z. Lu, and Y. Shi, "Time-dependent failure possibility analysis under consideration of fuzzy uncertainty," *Fuzzy Sets and Systems*, vol. 367, no. 15, pp. 19–35, 2019.
[9] C. Wang, Z. Qiu, M. Xu, and Y. Li, "Novel reliability-based optimization method for thermal structure with hybrid random, interval and fuzzy parameters," *Applied Mathematical Modelling*, vol. 47, pp. 573–586, 2017.
[10] C. Wang, Z. Qiu, M. Xu, and H. Qiu, "Novel fuzzy reliability analysis for heat transfer system based on interval ranking method," *International Journal of Thermal Sciences*, vol. 116, pp. 234–241, 2017.
[11] J. Cai, W. Huang, and H. Cheng, "Solving a fully fuzzy linear programming problem through compromise programming," *Journal of Applied Mathematics*, vol. 2013, no. 4, pp. 972–991, 2013.

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems

[12] H. Saberi Najafi and S. Edalatpanah, "A note on "A new method for solving fully fuzzy linear programming problems"," *Applied Mathematical Modelling*, vol. 37, no. 14–15, pp. 7865–7867, 2013.
[13] Z. Tang, "Solving Stackelberg equilibrium for multi objective aerodynamic shape optimization," *Applied Mathematical Modelling*, vol. 72, pp. 588–600, 2019.
[14] Z. Tang and L. Zhang, "A new Nash optimization method based on alternate elitist information exchange for multi-objective aerodynamic shape design," *Applied Mathematical Modelling*, vol. 68, pp. 244–266, 2019.
[15] E. Ghotbi, W. A. Otieno, and A. K. Dhingra, "Determination of Stackelberg-Nash equilibria using a sensitivity based approach," *Applied Mathematical Modelling*, vol. 38, no. 21-22, pp. 4972–4984, 2014.
[16] H. Yin, Y.-H. Chen, D. Yu, and H. Lü, "Nash-game-oriented optimal design in controlling fuzzy dynamical systems," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 8, pp. 1659–1673, 2018.
[17] J. Rosenhead, M. Elton, and S. K. Gupta, "Robustness and optimality as criteria for strategic decisions," *Journal of the Operational Research Society*, vol. 23, no. 4, pp. 413–431, 1972.
[18] H. Yin, D. Yu, S. Yin, and B. Xia, "Possibility-based robust design optimization for the structural-acoustic system with fuzzy parameters," *Mechanical Systems and Signal Processing*, vol. 102, pp. 329–345, 2018.
[19] Z.-C. Tang, Z.-Z. Lu, and J.-X. Hu, "An efficient approach for design optimization of structures involving fuzzy variables," *Fuzzy Sets and Systems*, vol. 255, pp. 52–73, 2014.
[20] Q. Da and X. Liu, "A satisfactory solution for interval number linear programming," *Journal of Systems Engineering*, vol. 2, pp. 123–128, 1999.
[21] X.-G. Shang and W.-S. Jiang, "A note on fuzzy information measures," *Pattern Recognition Letters*, vol. 18, no. 5, pp. 425–432, 1997.
[22] G. Wei, "Picture fuzzy cross-entropy for multiple attribute decision making problems," *Journal of Business Economics and Management*, vol. 17, no. 4, pp. 491–502, 2016.
[23] A. Mahmoodi, "Stackelberg-Nash equilibrium of pricing and inventory decisions in duopoly supply chains using a nested evolutionary algorithm," *Applied Soft Computing*, vol. 86, p. 105922, 2020.
[24] E. D'Amato, E. Daniele, L. Mallozzi, G. Petrone, and S. Tancredi, "A hierarchical multimodal hybrid Stackelberg–Nash GA for a leader with multiple followers game," in *Dynamics of Information Systems: Mathematical Foundations*, pp. 267–280, Springer, 2012.
[25] X. Zhou, C. Yang, and W. Gui, "A statistical study on parameter selection of operators in continuous state transition algorithm," *IEEE Transactions on Cybernetics*, vol. 49, no. 10, pp. 3722–3730, 2019.
[26] J. Han, C. Yang, X. Zhou, and W. Gui, "A new multi-threshold image segmentation approach using state transition algorithm," *Applied Mathematical Modelling*, vol. 44, pp. 588–601, 2017.
[27] X. Zhou, M. Huang, T. Huang, C. Yang, and W. Gui, "Dynamic optimization for copper removal process with continuous production constraints," *IEEE Transactions on Industrial Informatics*, 2019. doi:10.1109/TII.2019.2943500.
[28] R. Meng, K. H. Cheong, W. Bao, K. K. L. Wong, L. Wang, and N. Xie, "Multi-objective optimization of an arch dam shape under static loads using an evolutionary game method," *Engineering Optimization*, vol. 50, no. 6, pp. 1061–1077, 2018.
[29] B. D. Youn, K. Choi, R.-J. Yang, and L. Gu, "Reliability-based design optimization for crashworthiness of vehicle side impact," *Structural and Multidisciplinary Optimization*, vol. 26, no. 3-4, pp. 272–283, 2004.
[30] H. Gu and X. Wang, "Application of NSGA-II algorithm in the design of car body lateral crashworthiness," in *Proceedings of the 5th International Conference on Materials Engineering for Advanced Technologies*, pp. 44–47, 2016.
[31] A. D. P. Dourado, F. S. Lobato, A. Ap Cavalini, and V. Steffen, "Fuzzy reliability-based optimization for engineering system design," *International Journal of Fuzzy Systems*, vol. 21, no. 5, pp. 1418–1429, 2019.
[32] J. Han, C. Yang, C.-C. Lim, X. Zhou, P. Shi, and W. Gui, "Power scheduling optimization under single-valued neutrosophic uncertainty," *Neurocomputing*, vol. 382, pp. 12–20, 2020.

Jie Han received the B.S. degree in automation from Central South University, Changsha, China, in 2014, and received the Ph.D. degree in Control Science and Engineering in 2020 from Central South University.

She is currently a lecture at Central South University, Changsha, China. Her main research interests include robust optimization, optimization under uncertainty, optimization and control of complex industrial process.

Chunhua Yang (M'09) received the B.S. and M.S. degrees in automatic control engineering and the Ph.D. degree in control science and engineering from Central South University, Changsha, China, in 1985, 1988, and 2002, respectively.

From 1999 to 2001, she was a Visiting Professor with the University of Leuven, Leuven, Belgium. Since 1999, she has been a Full Professor with the School of Information Science and Engineering, Central South University. From 2009 to 2010, she was a Senior Visiting Scholar with the University of Western Ontario, London, ON, Canada. Her current research interests include modeling and optimal control of complex industrial process, fault diagnosis, and intelligent control system.

Cheng-Chew Lim (M'82–SM'02) received the Ph.D. degree from Loughborough University, Leicestershire, U.K., in 1981. He is a Professor with the School of Electrical and Electronic Engineering, The University of Adelaide, Adelaide, Australia. His research interests include control systems, machine learning, wireless communications, and optimization techniques and applications.

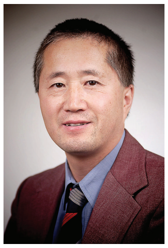
Dr. Lim serves as an Associate Editor of the IEEE Transactions on Systems, Man and Cybernetics: Systems, and an Editorial Board Member for the Journal of Industrial and Management Optimization and has served as a Guest Editor of a number of journals, including Discrete and Continuous Dynamical Systems Series.

Xiaojun Zhou (M'19) received his B.S. degree in Automation in 2009 from Central South University, Changsha, China and received the Ph.D. degree in Applied Mathematics in 2014 from Federation University Australia.

He is currently an associate professor at Central South University, Changsha, China. His main interests include modeling, optimization and control of complex industrial process, artificial intelligence, optimization theory and algorithms, duality theory, global optimization (both deterministic and stochastic, especially state transition algorithm), and their applications.

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3025697, IEEE Transactions on Fuzzy Systems



Peng Shi (M'95–SM'98–F'15) received the PhD degree in Electrical Engineering from the University of Newcastle, Australia in 1994. He was awarded the Doctor of Science degree from the University of Glamorgan, Wales in 2006; and the Doctor of Engineering degree from the University of Adelaide, Australia in 2015.

He is now a professor at the University of Adelaide. His research interests include system and control, autonomous and robotic systems, network and cyber-physical systems. He is now a Member of Board of Governors, IEEE SMC Society, and an IEEE Distinguished Lecturer. He is a Fellow of the Institution of Engineering and Technology; and the Institute of Engineers, Australia. He has served on the editorial board of a number of journals, including Automatica; IEEE Transactions on (Automatic Control; Cybernetics; Fuzzy Systems; Circuits and Systems).

1063-6706 (c) 2020 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Authorized licensed use limited to: Central South University. Downloaded on May 11,2021 at 12:57:51 UTC from IEEE Xplore. Restrictions apply.